

Jens Eggers

Singularities: Formation, Structure, and Propagation



J. EGERS AND
M. A. FONTELOS

CAMBRIDGE TEXTS
IN APPLIED
MATHEMATICS

Eggers
Fontelos

Singularities: Formation,
Structure, and Propagation

Many key phenomena in physics and engineering are described as singularities in the solutions to the differential equations describing them. Examples covered thoroughly in this book include the formation of drops and bubbles, the propagation of a crack, and the formation of a shock in a gas.

Aimed at a broad audience, this book provides the mathematical tools for understanding singularities and explains the many common features in their mathematical structure. Part I introduces the main concepts and techniques, using the most elementary mathematics possible so that it can be followed by readers with only a general background in differential equations. Parts II and III require more specialized methods of partial differential equations, complex analysis, and asymptotic techniques. The book may be used for advanced fluid mechanics courses and as a complement to a general course on applied partial differential equations.

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The aim of this series is to provide a focus for publishing textbooks in applied mathematics at the advanced undergraduate and beginning graduate level. The books are devoted to covering certain mathematical techniques and theories and exploring their applications.

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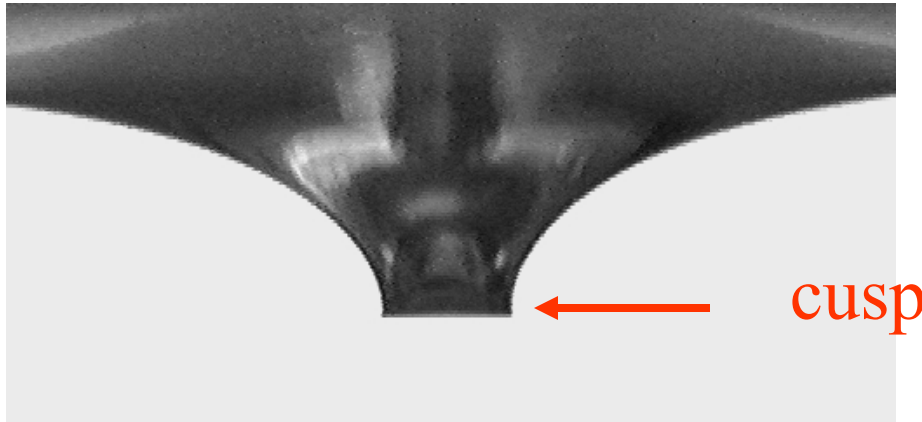
Cover illustrations courtesy of Nick Lane and David Bates

Geometric singularities



University of
BRISTOL

Cusp universality



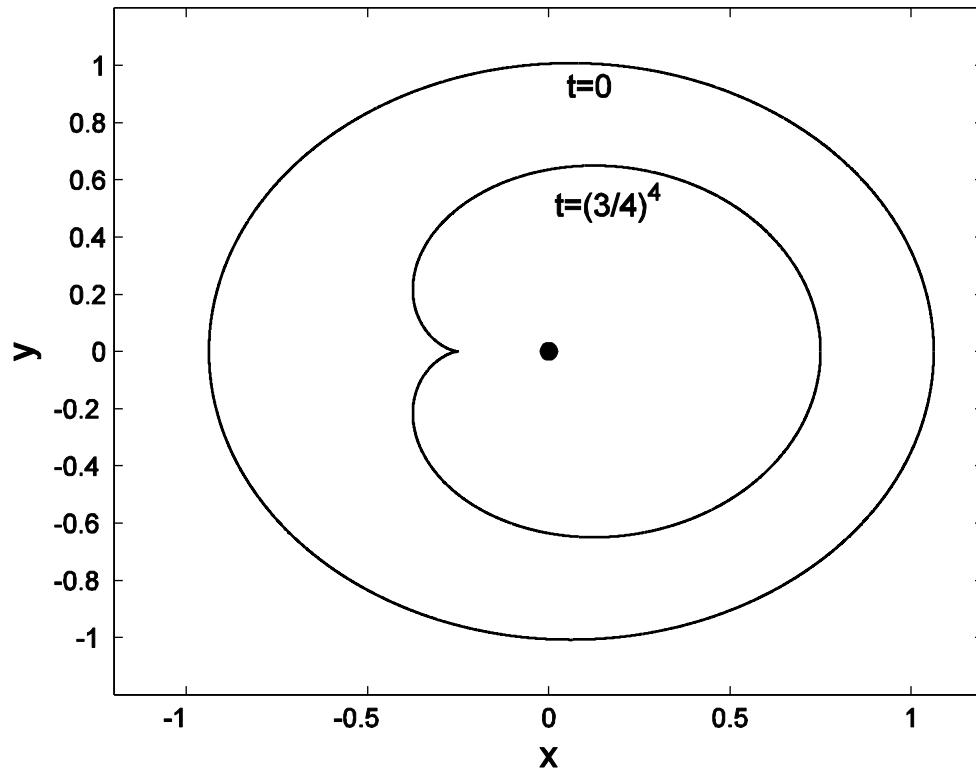
surface of a viscous
fluid

width : $r^{3/2}$



caustics in a
cup

Hele-Shaw cell

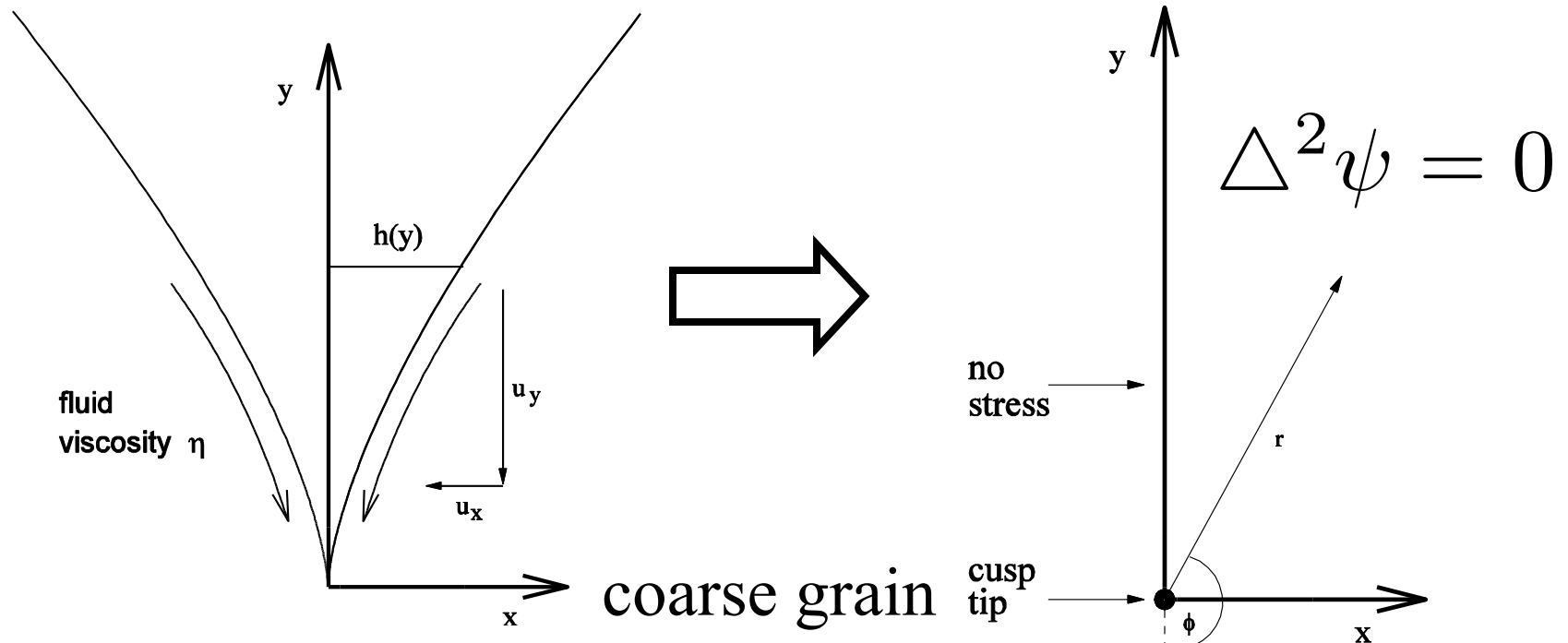


bubble with
sink in center

Polubarinova-Kochina, 1945

Galin, 1945

Cusp structure: viscous flow



similarity solution: $\psi = r^\alpha f(\phi)$

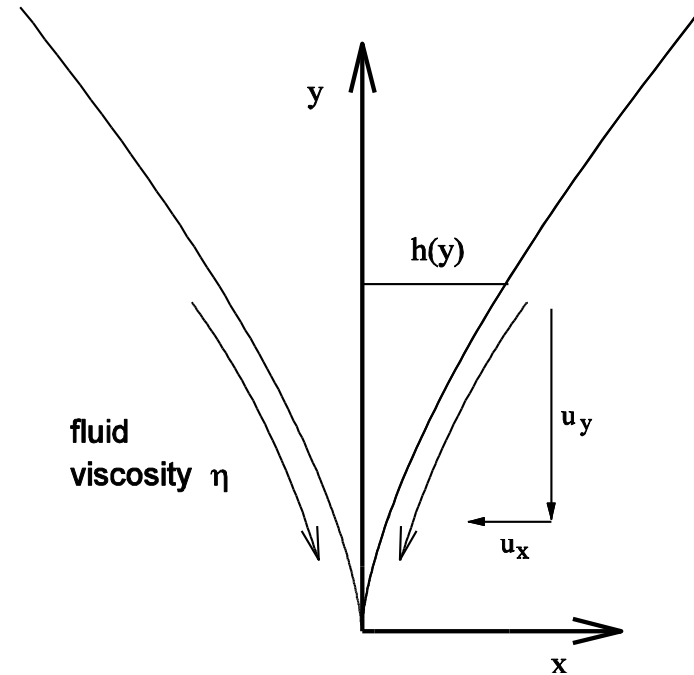
$$\alpha = 1, 3/2, 2, \dots$$

Local analysis

$$u_x = -6Ay^{1/2}, \quad u_y = -V$$

$$\frac{\partial h}{\partial y} = \frac{u_x}{u_y} = \frac{6A}{V} y^{1/2}$$

$$h : y^{3/2}$$



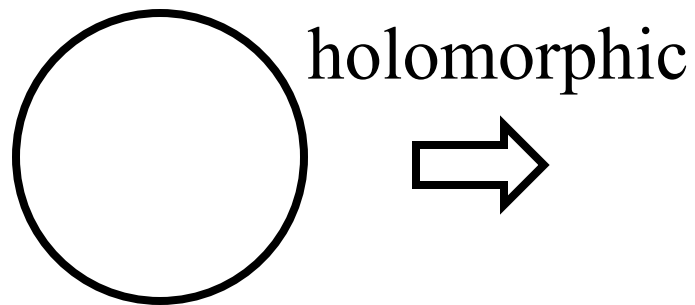
$$\psi = Vx$$

$$-4Ar^{3/2} \sin^3 \frac{\phi}{2}$$

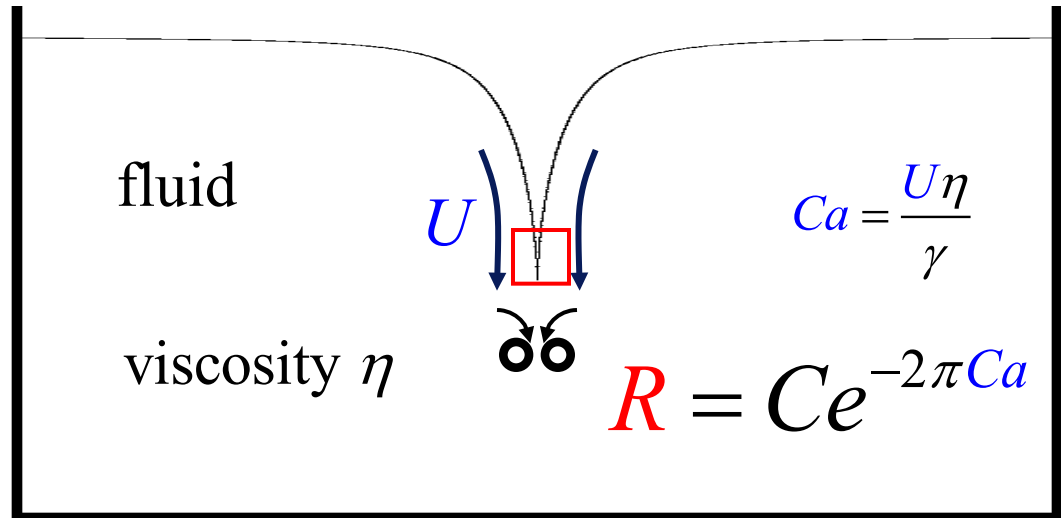
Renardy et al., JFM '91

Jeong and Moffatt solution

J.-T. Jeong, H.K. Moffatt, JFM '92

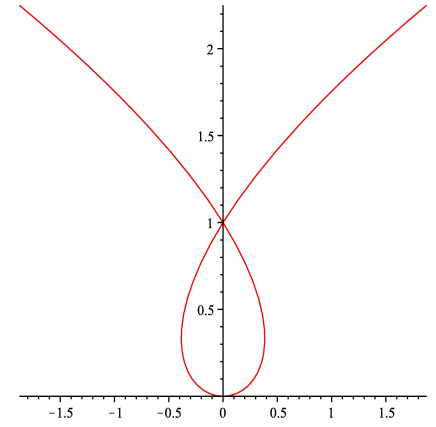
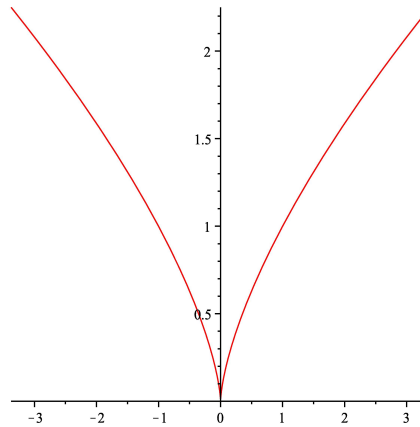
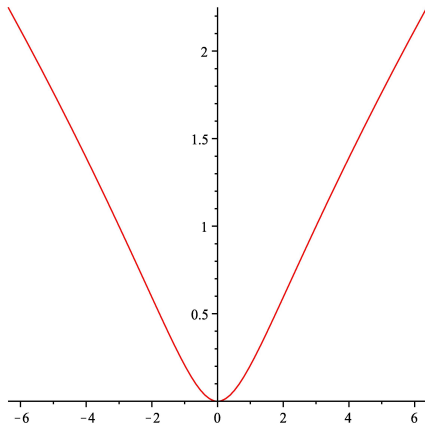


$z(\theta)$ smooth



singularity: $z'(0) = 0$

Cusp geometry



critical point: $a_1 = 0$

$$x = a_1 \varphi$$

$$y = 0$$

$$x = \epsilon X, \quad y = \epsilon^{3/2} Y$$

$$X = \frac{\sigma^2}{2}, \quad Y = \frac{\sigma^3}{3} \pm \sigma$$

universal $x = \varepsilon \varphi + a \varphi^3 / 3$

unfolding: $y = \varphi^2 / 2$

Singularity theory

left-right equivalent: $g = \psi \circ f \circ \phi^{-1}$

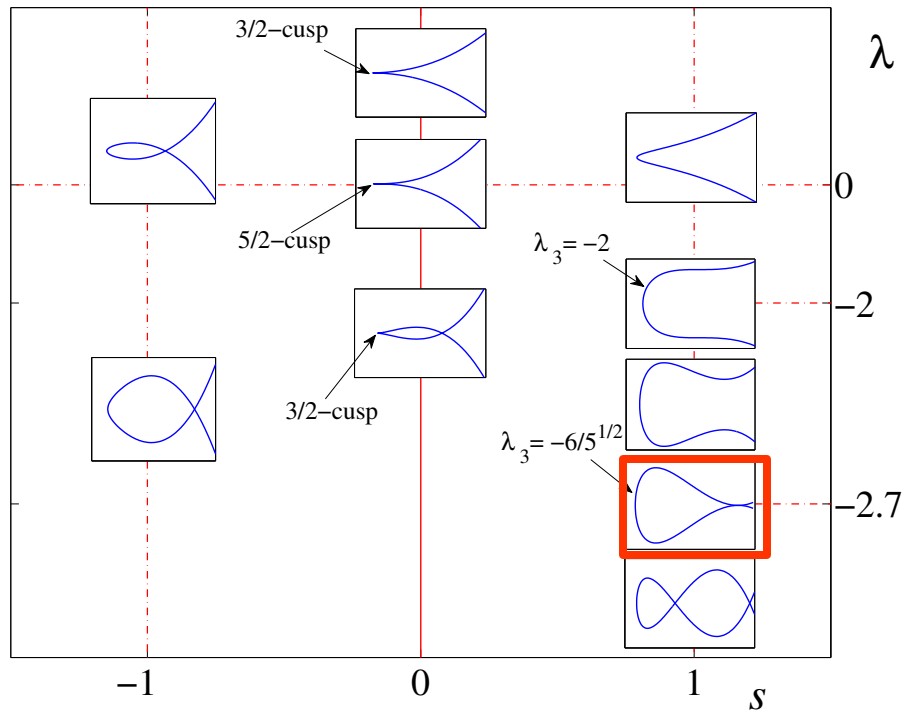
singular germ: $\text{rk}_0(f) < \min(n, p)$

unfolding: $F(\mathbf{x}, \mathbf{u}), \quad F(\mathbf{x}, \mathbf{u} = 0) = f(\mathbf{x})$

plane curves: $(\varphi^m, \varphi^n), \quad \text{hcf}(m, n) = 1$

Example: $(\underbrace{\varphi^2, \varphi^5}_{\text{germ}} + \underbrace{\mu_1 \varphi + \mu_3 \varphi^3}_{\text{unfolding}})$

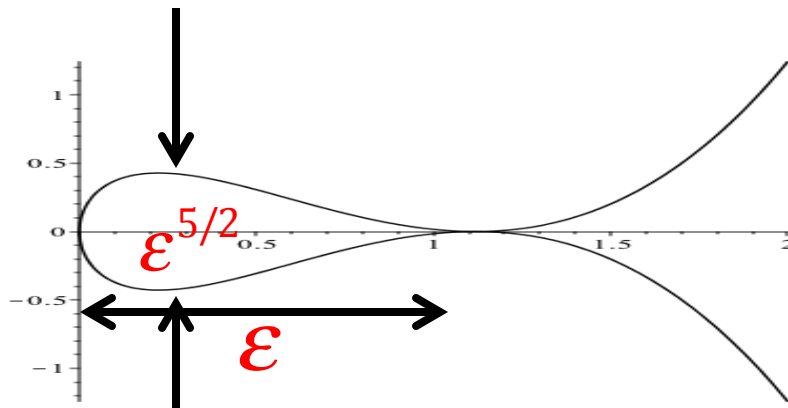
A small bubble



$$X = \frac{\sigma^2}{2}$$

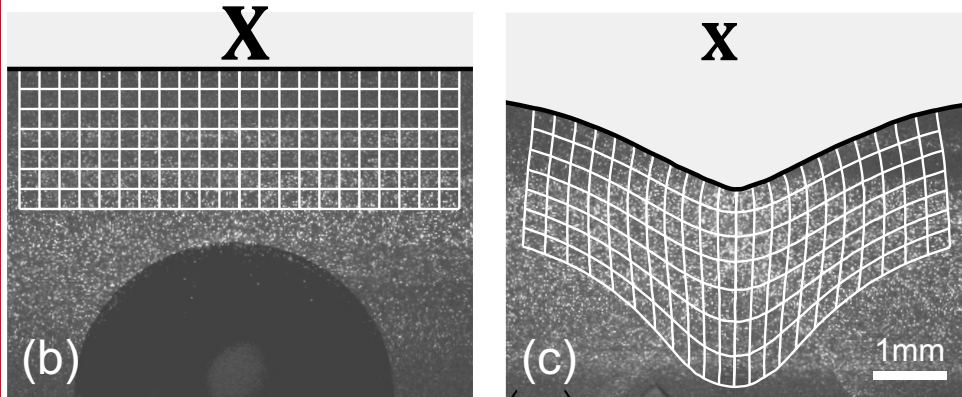
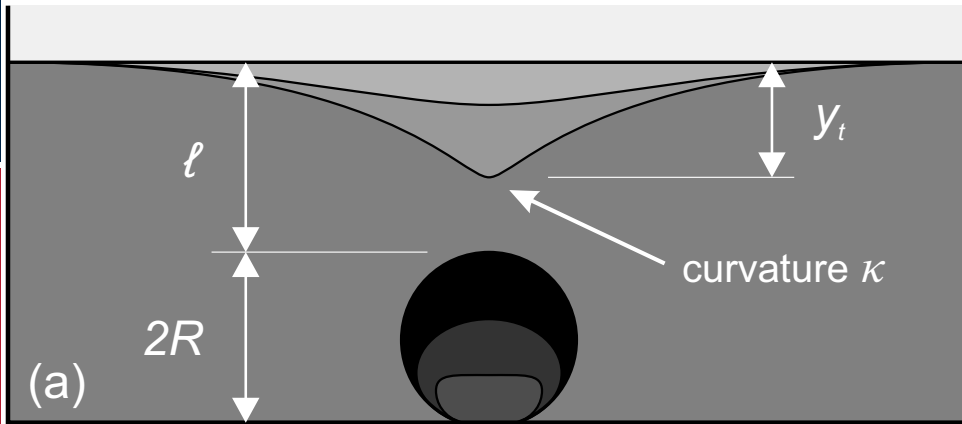
$$Y = \sigma \left(\frac{\sigma^4}{5} + \frac{\lambda_3}{3} \sigma^2 + s \right)$$

$$(X, Y) = \left(\frac{\sigma^2}{2}, \frac{\sigma}{5} \left(\sigma^2 - \sqrt{5} \right)^2 \right)$$



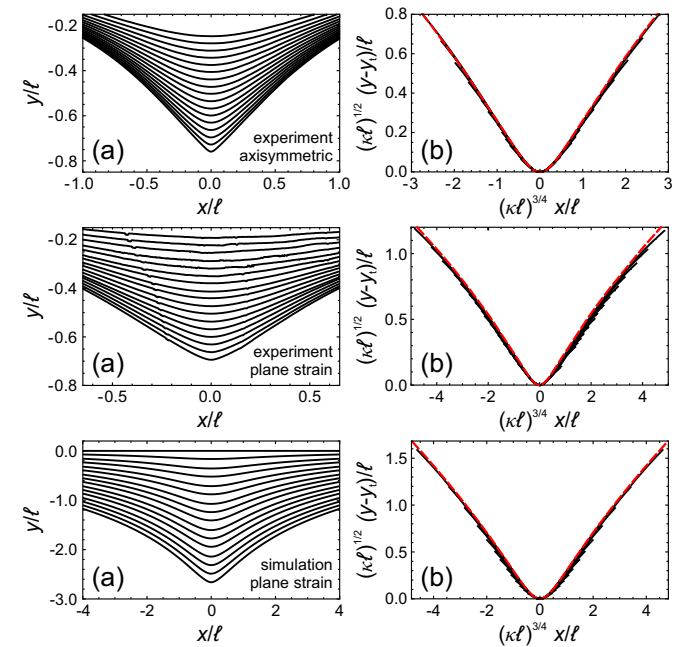
Elastic cusp

with S. Karpitschka,
J. Snoeijer

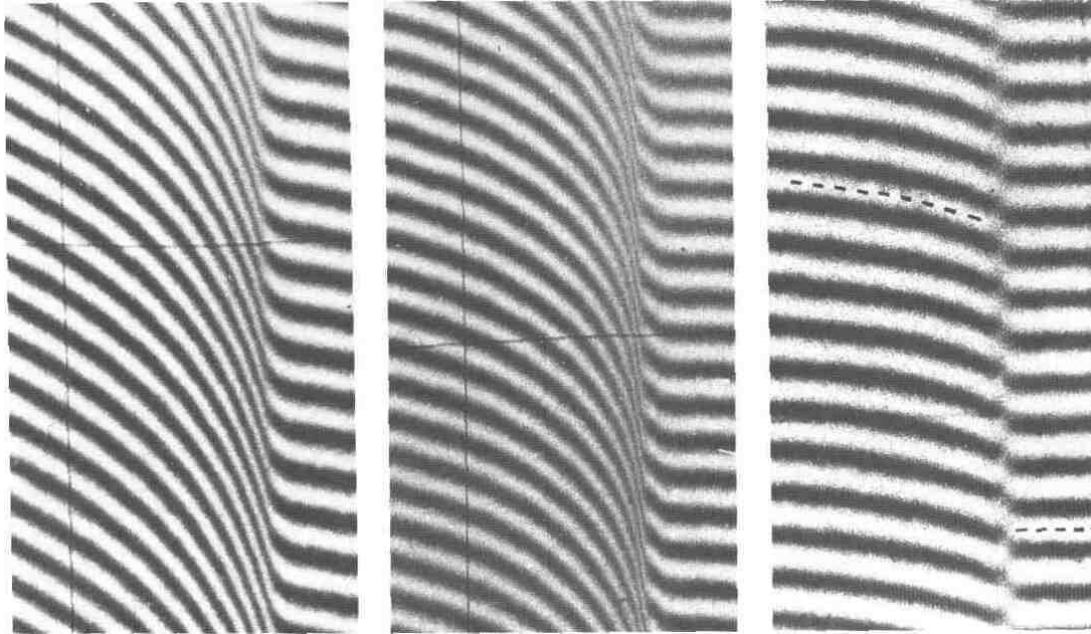


$$\mathbf{x} = f(\mathbf{X})$$

$$x = \kappa^3 \left(s + as^3 \right) \quad y = \kappa^2 s^2$$



A shock wave



a jump in density occurs at some
finite time t_0 !

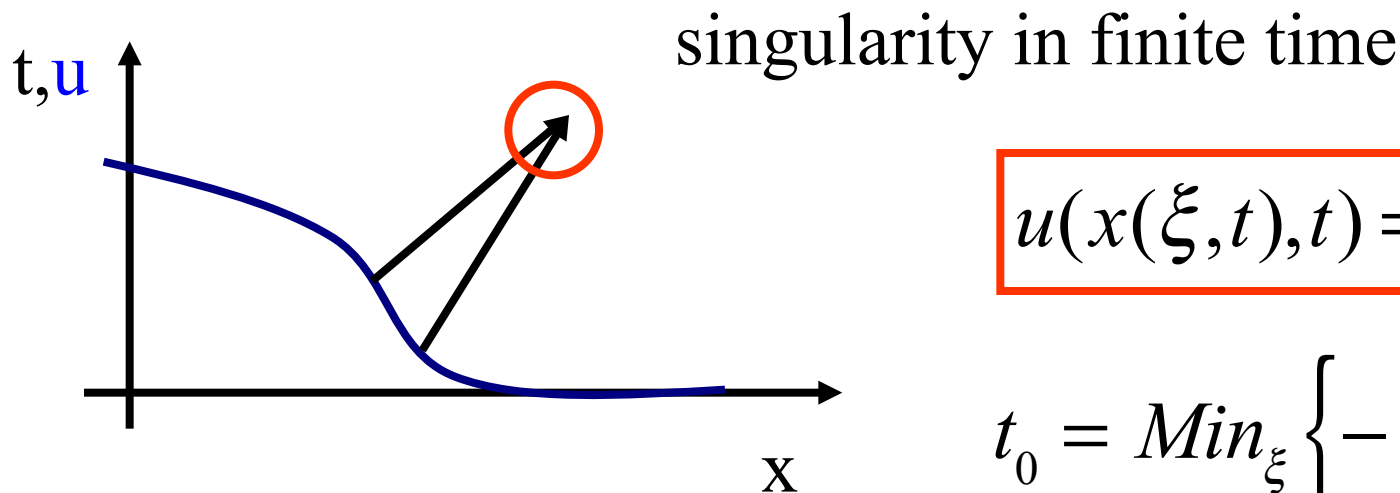
W.C. Griffith, W. Bleakney

Burgers' equation

characteristic curves:

$$x(\xi, t) = u_0(\xi)t + \xi$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$$



$$u(x(\xi, t), t) = u_0(\xi)$$

$$t_0 = \text{Min}_{\xi} \left\{ -1/u_0'(\xi) \right\}$$

$$u_0(\xi) = -s' + as'^3 + \dots$$

Similarity solution $t' = t_0 - t$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$$

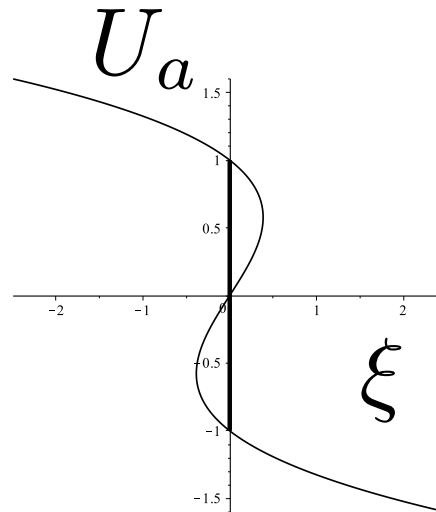
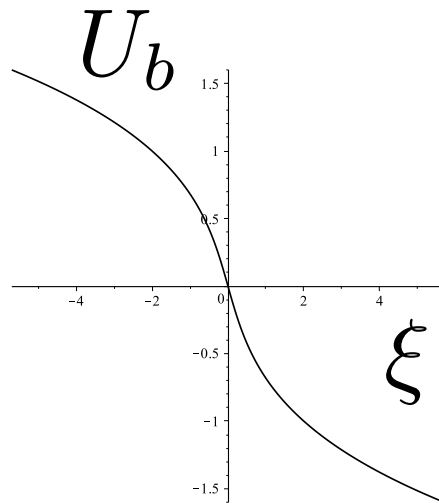
$$u = t'^{\alpha} U \left(\frac{x}{t'^{\alpha+1}} \right) \quad \xi = x/t'^{\alpha+1}$$

$$-\alpha U + (1 + \alpha) \xi U' + U U' = 0$$

$$\xi = \begin{cases} -U - C U^{1+1/\alpha}, & \alpha_i = \frac{1}{2i+2}, i = 0, 1, 2, \dots \\ -U, & \alpha = 0 \end{cases} \quad \begin{array}{l} \text{regular at} \\ \xi = 0 \end{array}$$

Similarity solution

$$t' = t_0 - t$$



$$\xi + U_b + CU_b^3 = 0$$

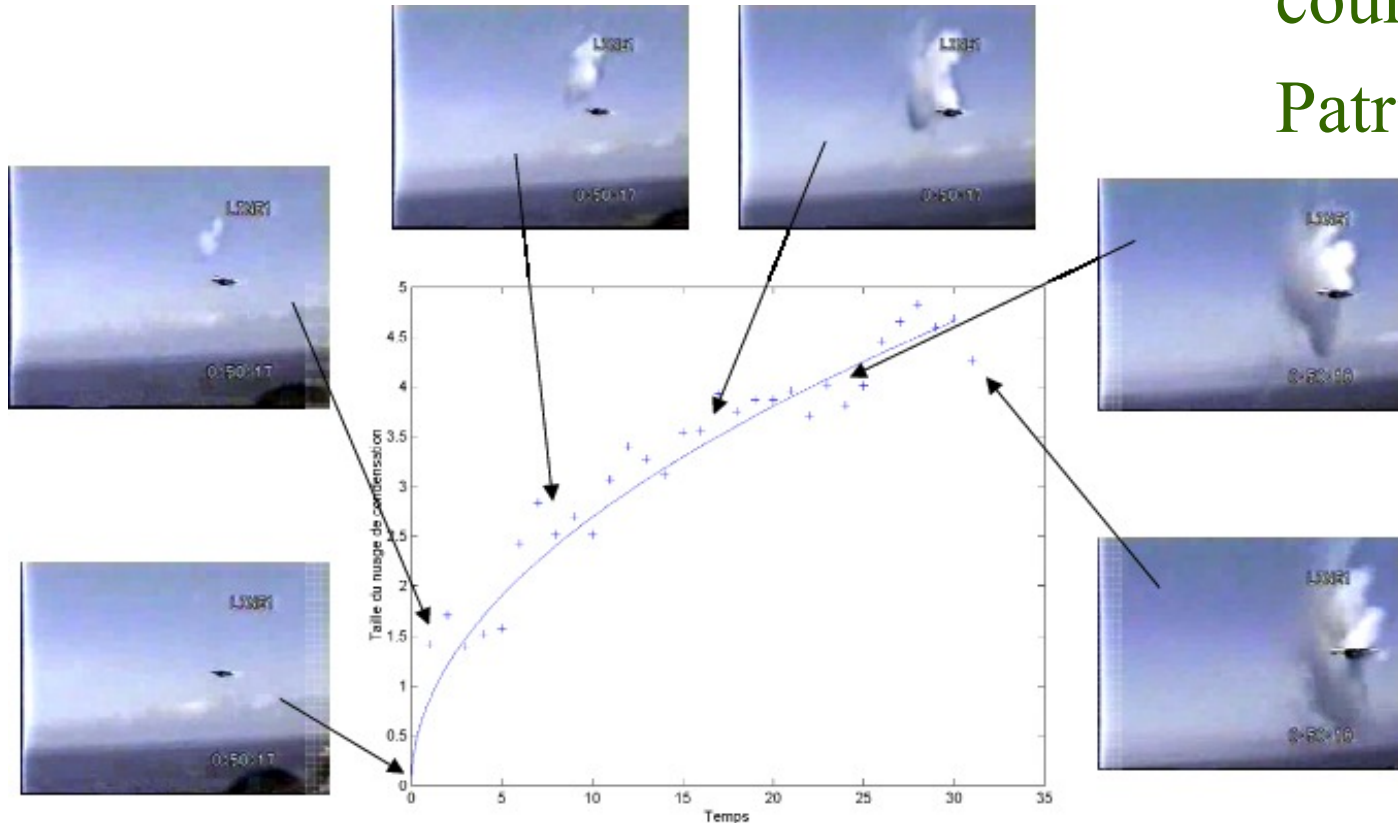
$$\xi - U_a + CU_a^3 = 0$$

$$u(x, t) = t'^{1/2} U(x / t'^{3/2})$$

only stable solution!

2D structure of shock waves

courtesy of
Patrice Legal



$$t_c(y) - t_0 = ay^2 + O(y^3), \quad a > 0, \quad y \sim t^{1/2}$$

Compressible Euler with T. Grava

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 & p &= \frac{A}{\gamma} \rho^\gamma \\ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} &= -\frac{1}{\rho} \nabla p & \mathbf{v} &= \nabla \phi\end{aligned}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 = -\frac{A}{\gamma - 1} \left(\rho^{\gamma-1} - \rho_0^{\gamma-1} \right)$$

$$|\nabla \rho|_{\rho=\rho_0} \rightarrow \infty$$

$$c_0^2 = \frac{\partial p}{\partial \rho} = A \rho_0^{\gamma-1}$$

$$\partial \rho / \partial y = 0, \quad \partial x / \partial \rho = 0$$

$$\partial^2 x / \partial \rho^2 = 0, \quad \frac{\partial^3 x}{\partial \rho^3} = \text{finite!}$$

Similarity solution

$$\phi = |t'|^2 \Phi(\xi, \eta), \quad \xi = \frac{x + c_0 t'}{|t'|^{3/2}}, \quad \eta = \frac{y}{|t'|^{1/2}}$$

$$\rho = \rho_0 [1 + |t'|^{1/2} R(\xi, \eta) + |t'| Q(\xi, \eta)]$$

$$-c_0 |t'|^{1/2} \Phi_\xi + |t'| [\mp 2\Phi \pm \frac{3\xi}{2} \Phi_\xi \pm \frac{\eta}{2} \Phi_\eta] + \frac{|t'|}{2} \Phi_\xi^2 =$$

In(B): $-\frac{c_0^2}{\gamma-1} \left\{ (\gamma-1) |t'|^{1/2} R + |t'| [(\gamma-1) Q + \frac{1}{2} (\gamma^2 - 3\gamma + 2) R^2] \right\} + O(t'^{3/2})$

In(C): $-c_0 R_\xi |t'|^{-1} + |t'|^{-1/2} [\mp \frac{R}{2} \pm \frac{3\xi}{2} R_\xi \pm \frac{\eta}{2} R_\eta - c_0 Q_\xi] +$
 $\rho_0 |t'|^{-1} \Phi_{\xi\xi} + |t'|^{-1/2} [\Phi_\xi R_\xi + \Phi_{\xi\xi} R] = O(t'^0)$

$$U = \Phi_\xi$$

$$U - 3\xi U_\xi - \eta U_\eta = \pm(\gamma + 1) U U_\xi$$

Similarity solution

$$\xi(U, \eta) : \xi_U U - 3\xi + \eta \xi_\eta = \pm(\gamma + 1)U$$

$$\xi = \mp \frac{\gamma + 1}{2} U - U^3 F\left(\frac{\eta}{U}\right)$$

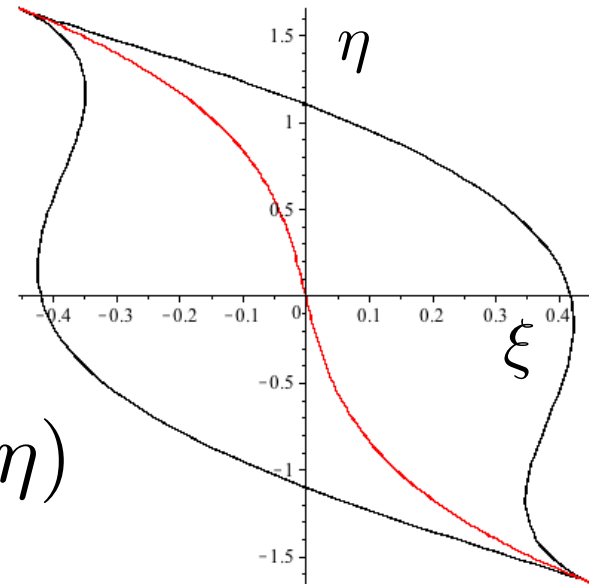
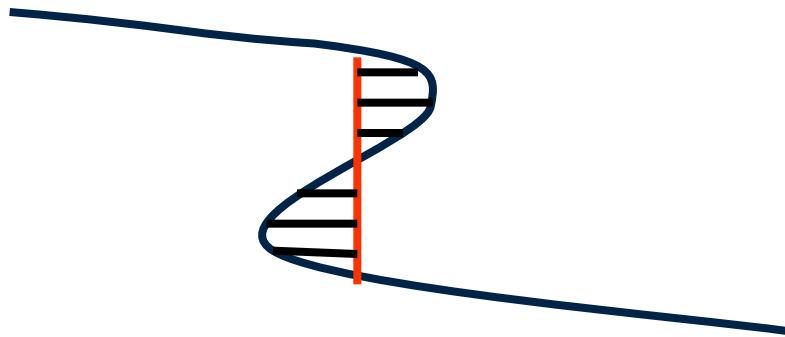
$$\text{regularity condition: } \left. \frac{\partial^3 \xi}{\partial \eta^3} \right|_0 = -F'''(a) \stackrel{!}{=} \text{const}$$

$$\xi = \frac{\gamma + 1}{2} U - A_0 U^3 - A_1 U^2 \eta - A_2 U \eta^2 - A_3 \eta^3$$

Shock position

$$\xi = \frac{\gamma + 1}{2} U - A_0 U^3 - A_1 U^2 \eta - A_2 U \eta^2 - A_3 \eta^3$$

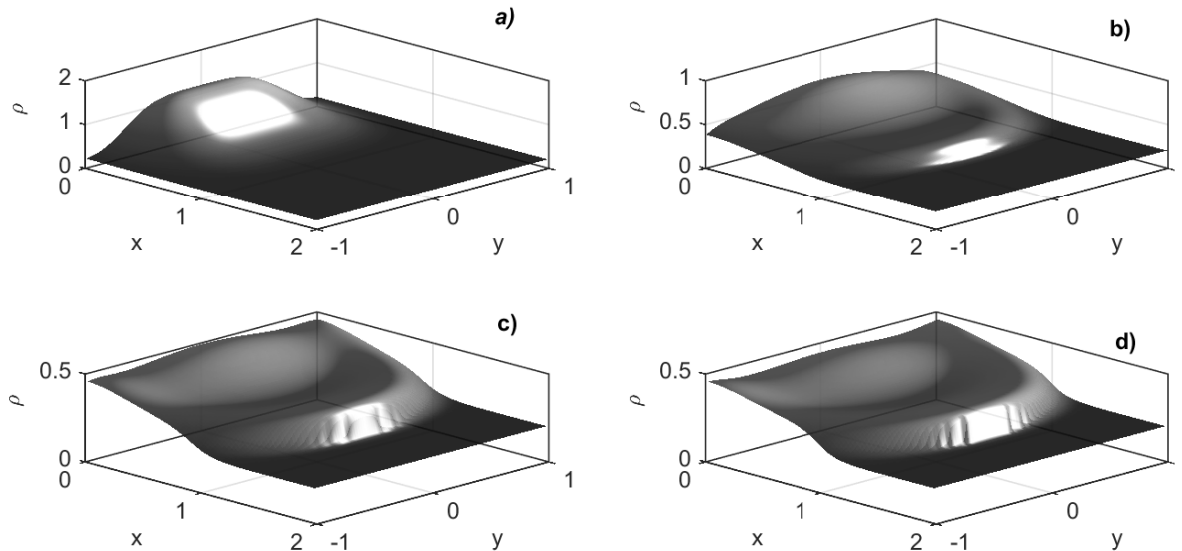
$$U(\xi, \eta) \text{ vertical: } \frac{\gamma + 1}{2} = 3A_0 U^2 + 2A_1 U \eta + A_2 \eta^2$$



$$\bar{\xi} = \xi - \xi_s(\eta), \quad \bar{U} = U - U_s(\eta)$$

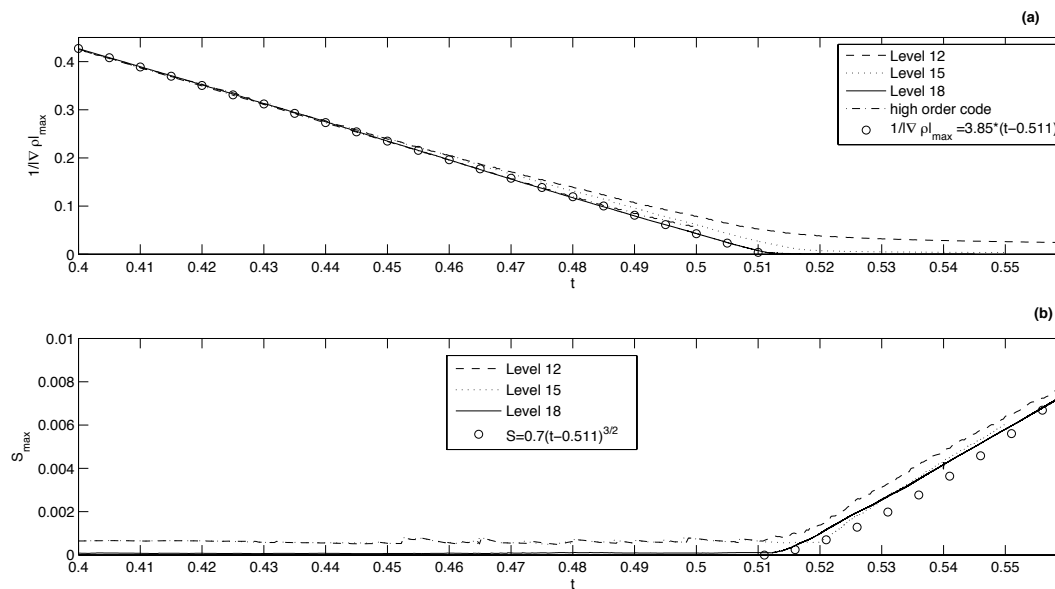
$$\bar{\xi} = -A_0 \bar{U} (\bar{U}^2 - \Delta^2(\eta))$$

Numerical simulation

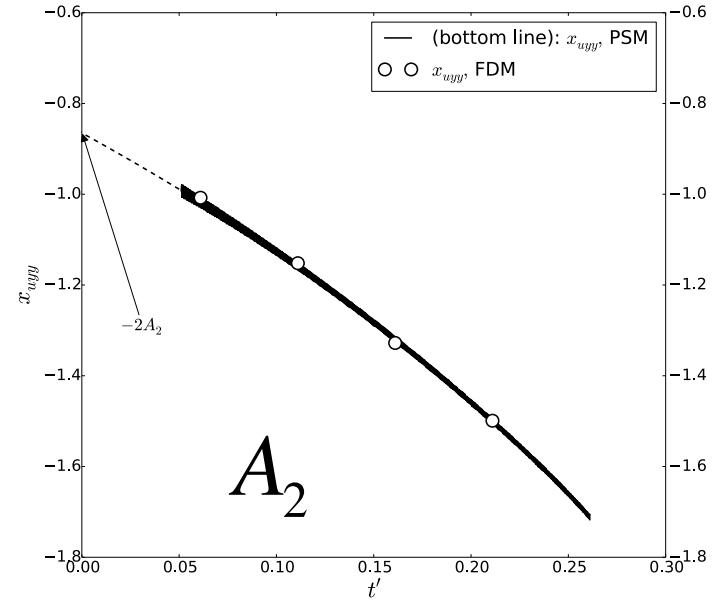
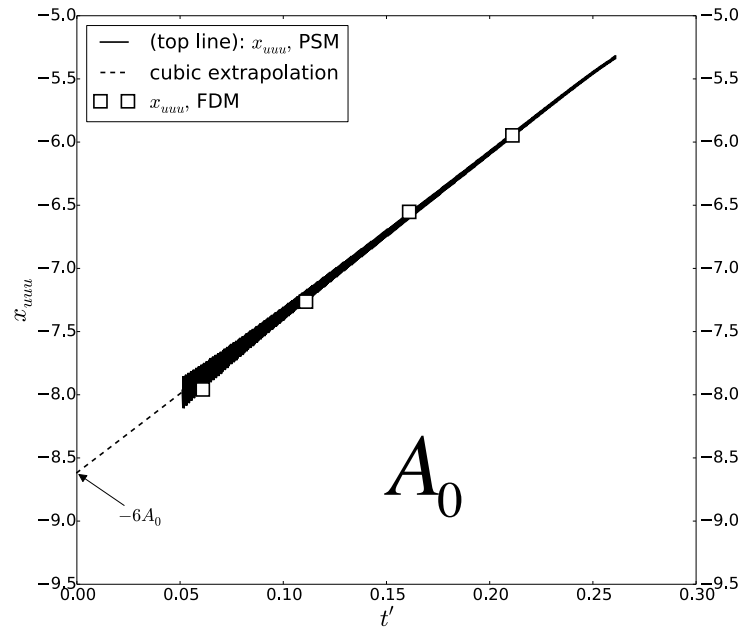


M.A. Herrada,
G. Pitton

with Basilisk



Parameters

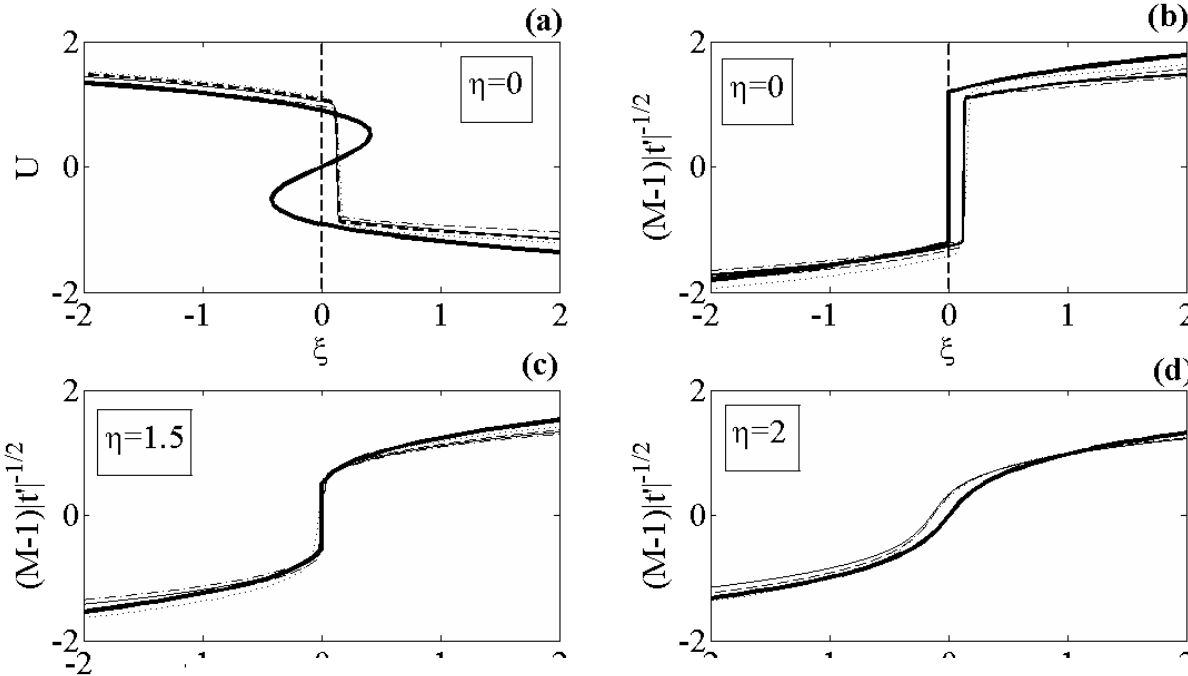


before singularity

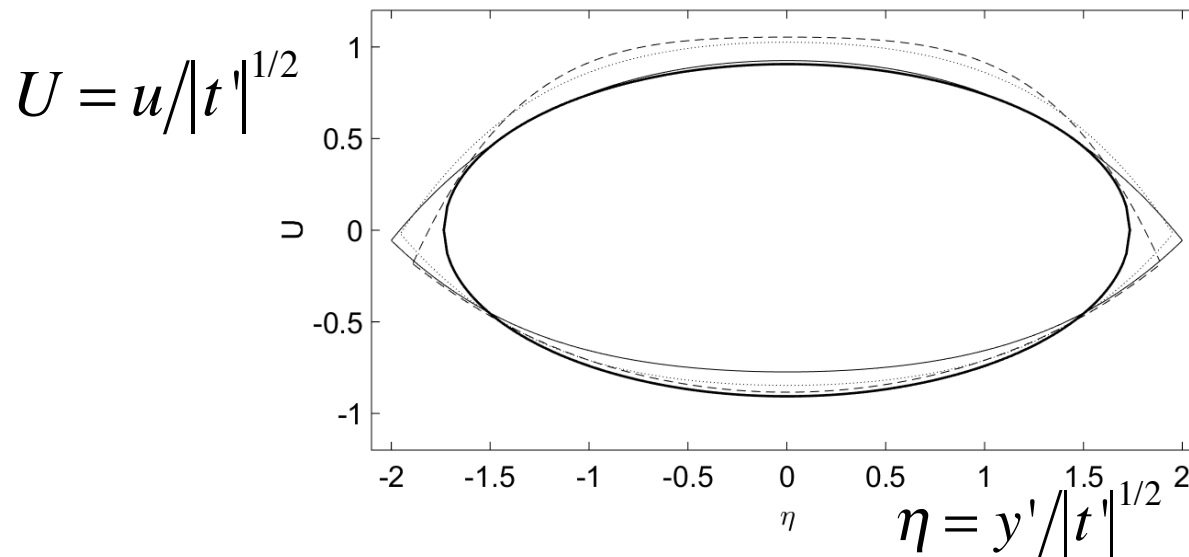
$$\xi = \frac{\gamma + 1}{2}U - A_0U^3 - A_1U^2\eta - A_2U\eta^2 - A_3\eta^3$$

Predictions

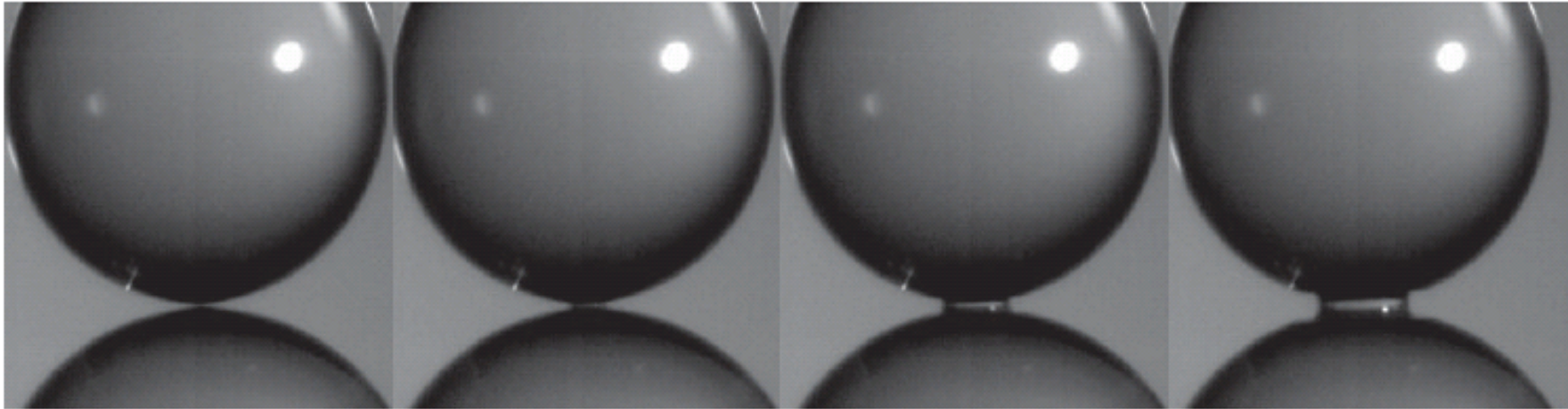
after singularity



$$\xi = (x' - c|t'|)/|t'|^{3/2}$$



Counterexample: drop coalescence



Aarts et al., PRL '05

width : r^2