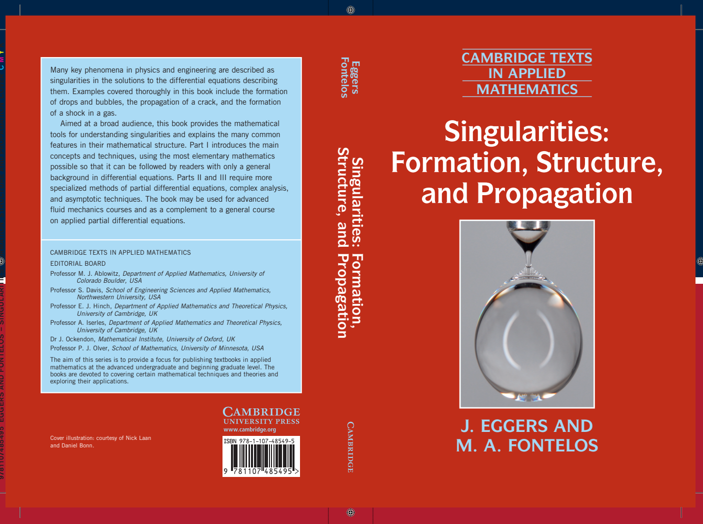

Jens Eggers

Geometric singularities

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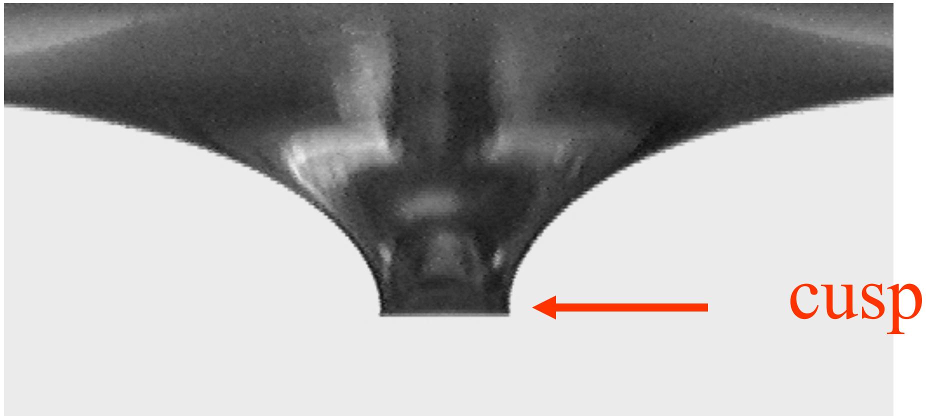
Cusp universality



surface of a viscous fluid

$$\text{width} : r^{3/2}$$

caustics in a cup

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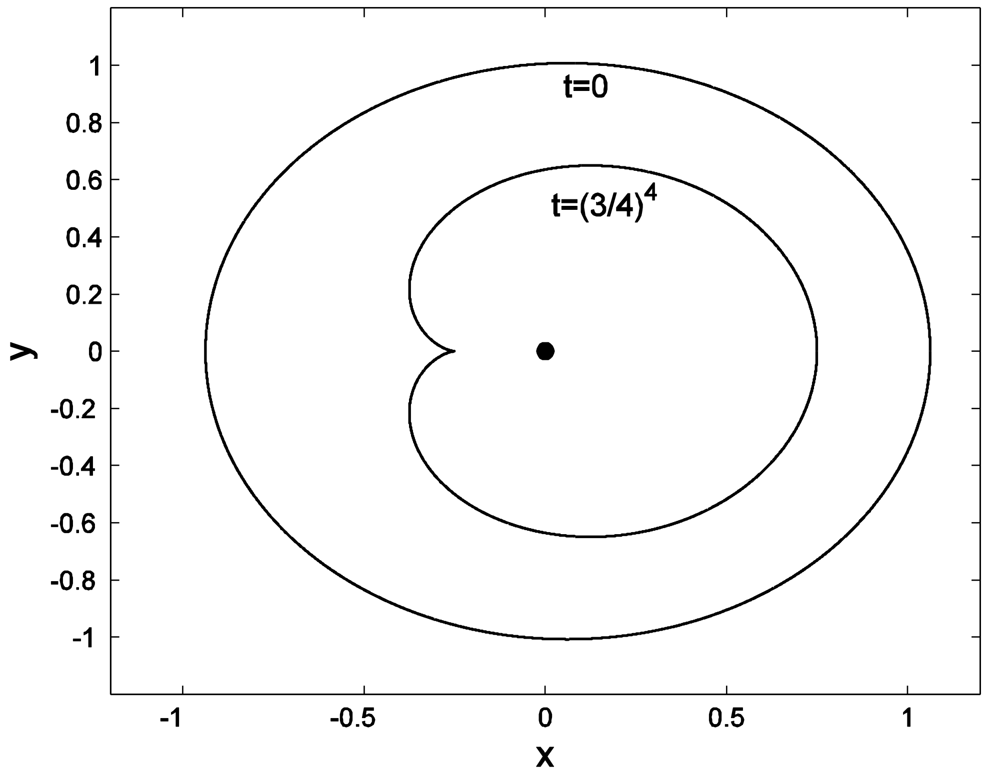
Hele-Shaw cell



bubble with sink in center

Polubarinova-Kochina, 1945

Galin, 1945

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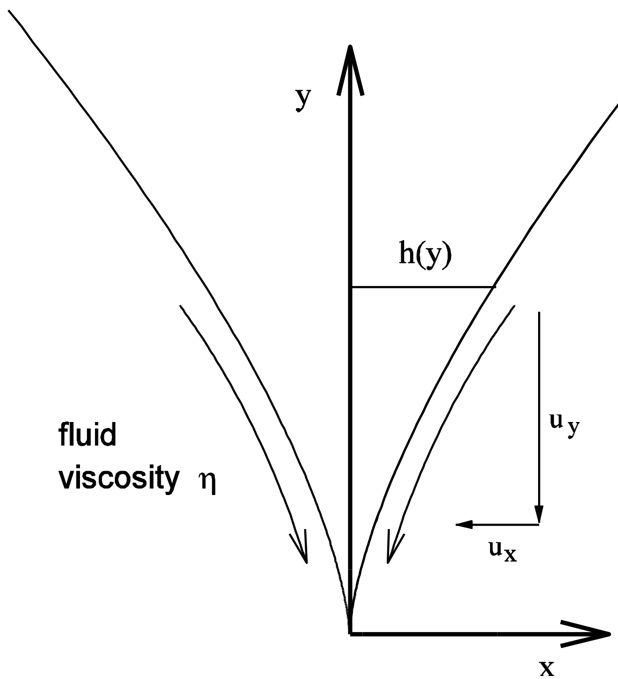
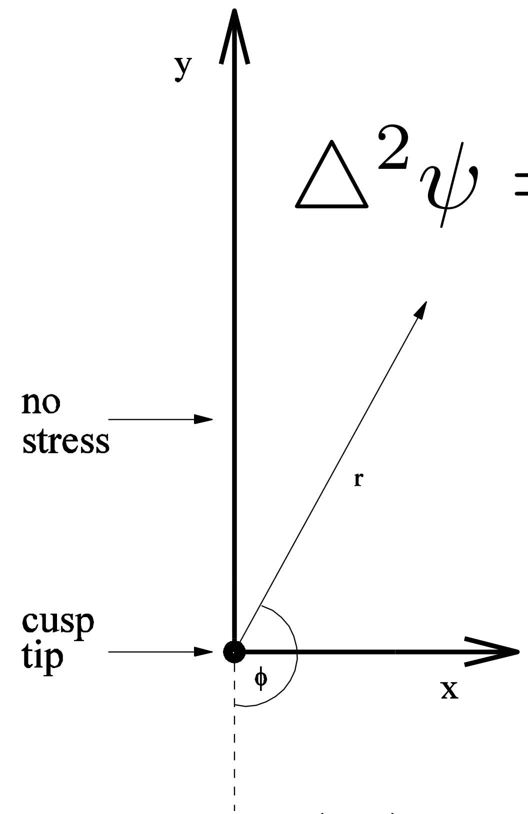
Cusp structure: viscous flow



coarse grain



$$\triangle^2 \psi = 0$$

similarity solution: $\psi = r^{\alpha} f(\phi)$

$$\alpha = 1, 3/2, 2, \ldots$$

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Local analysis

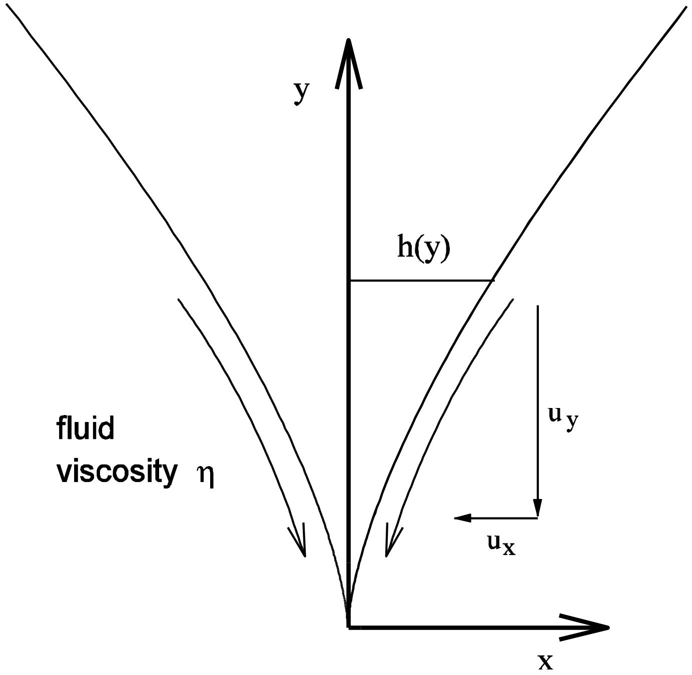
$$u_x = -6Ay^{1/2}, \quad u_y = -V$$

$$\frac{\partial h}{\partial y} = \frac{u_x}{u_y} = \frac{6A}{V} y^{1/2}$$

$$h : y^{3/2}$$

$$\psi = Vx \quad -4Ar^{3/2} \sin^3 \frac{\phi}{2}$$

Renardy et al., JFM `91



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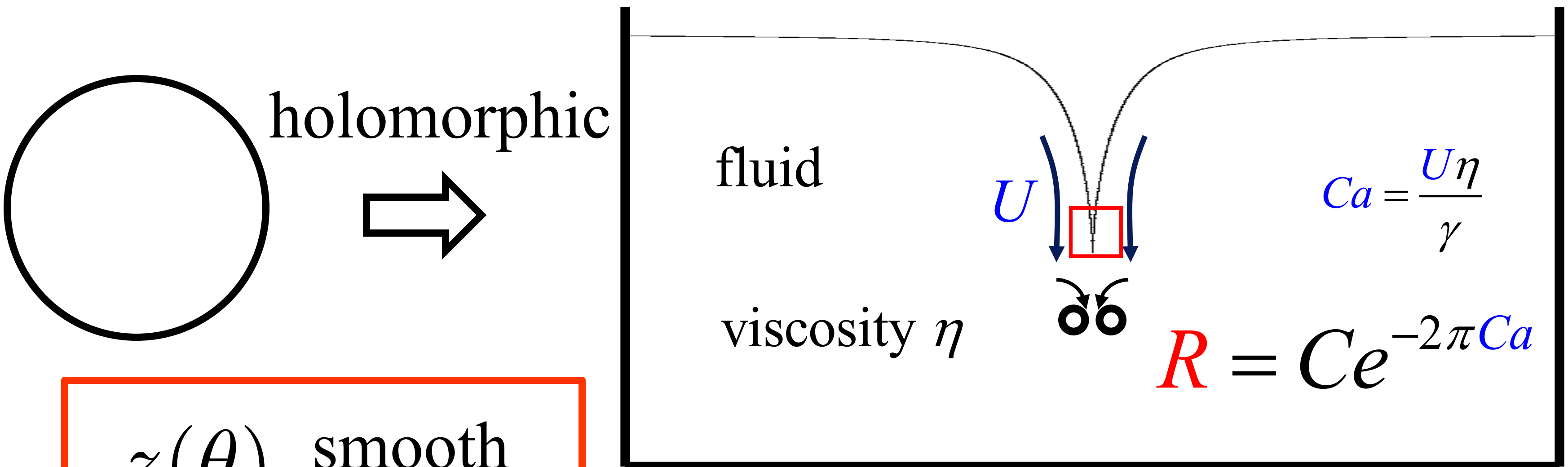
Jeong and Moffatt solution

J.-T. Jeong, H.K. Moffatt, JFM `92



$\boxed{z(\theta) \text{ smooth}}$

singularity: $z'(0) = 0$

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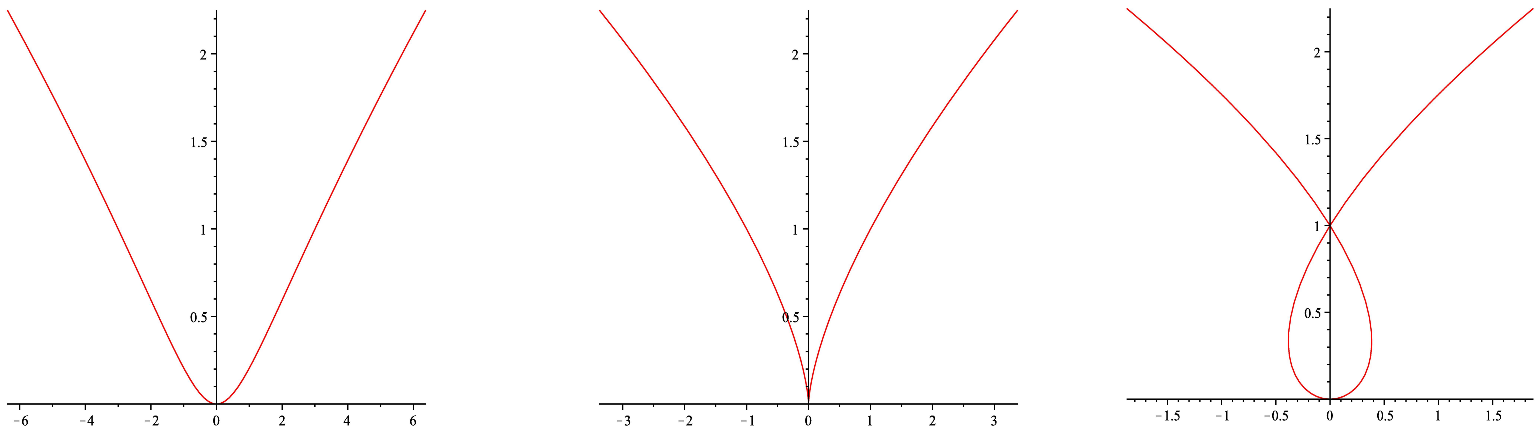
Cusp geometry



critical point: $a_1 = 0$

$$x = a_1\varphi$$

$$y = 0$$

$$x = \epsilon X, \quad y = \epsilon^{3/2} Y$$

$$X = \frac{\sigma^2}{2}, \quad Y = \frac{\sigma^3}{3} \pm \sigma$$

universal unfolding:

$$x = \varepsilon\varphi + a\varphi^3 / 3$$

$$y = \varphi^2 / 2$$

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Singularity theory

left-right equivalent: $g = \psi \circ f \circ \phi^{-1}$

singular germ: $\mathrm{rk}_0(f) < \min(n, p)$

unfolding: $F(\mathbf{x}, \mathbf{u}), \quad F(\mathbf{x}, \mathbf{u} = 0) = f(\mathbf{x})$

plane curves: $(\varphi^m, \varphi^n), \quad \mathrm{hcf}(\mathrm{m}, \mathrm{n}) = 1$

Example: $(\underbrace{\varphi^2, \varphi^5}_{\text{germ}} + \underbrace{\mu_1 \varphi + \mu_3 \varphi^3}_{\text{unufolding}})$

Eggers, Suramlishvili, Eur. J. Mech. B, 2017

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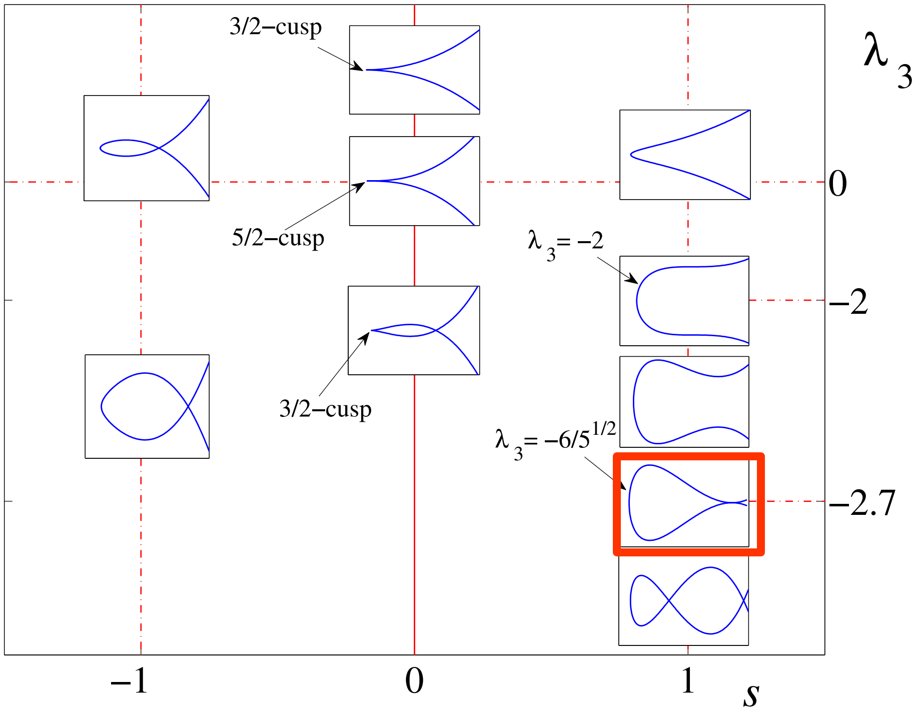
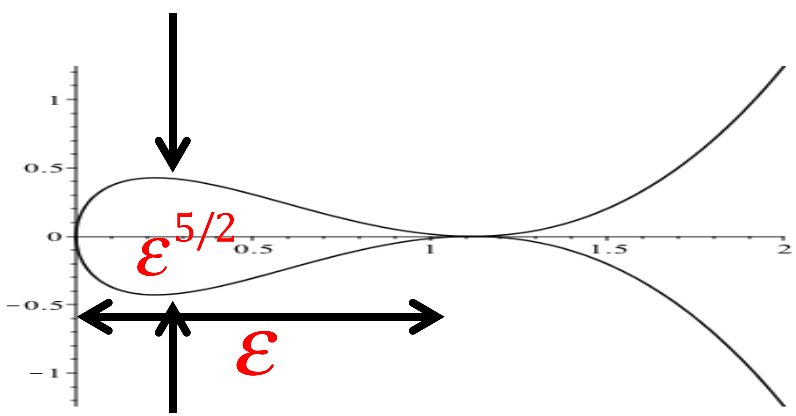
A small bubble



$$X = \frac{\sigma^2}{2}$$

$$Y = \sigma\left(\frac{\sigma^4}{5} + \frac{\lambda_3}{3}\sigma^2 + s\right)$$

$$\boxed{(X, Y) = \left(\frac{\sigma^2}{2}, \frac{\sigma}{5}\left(\sigma^2 - \sqrt{5}\right)^2\right)}$$



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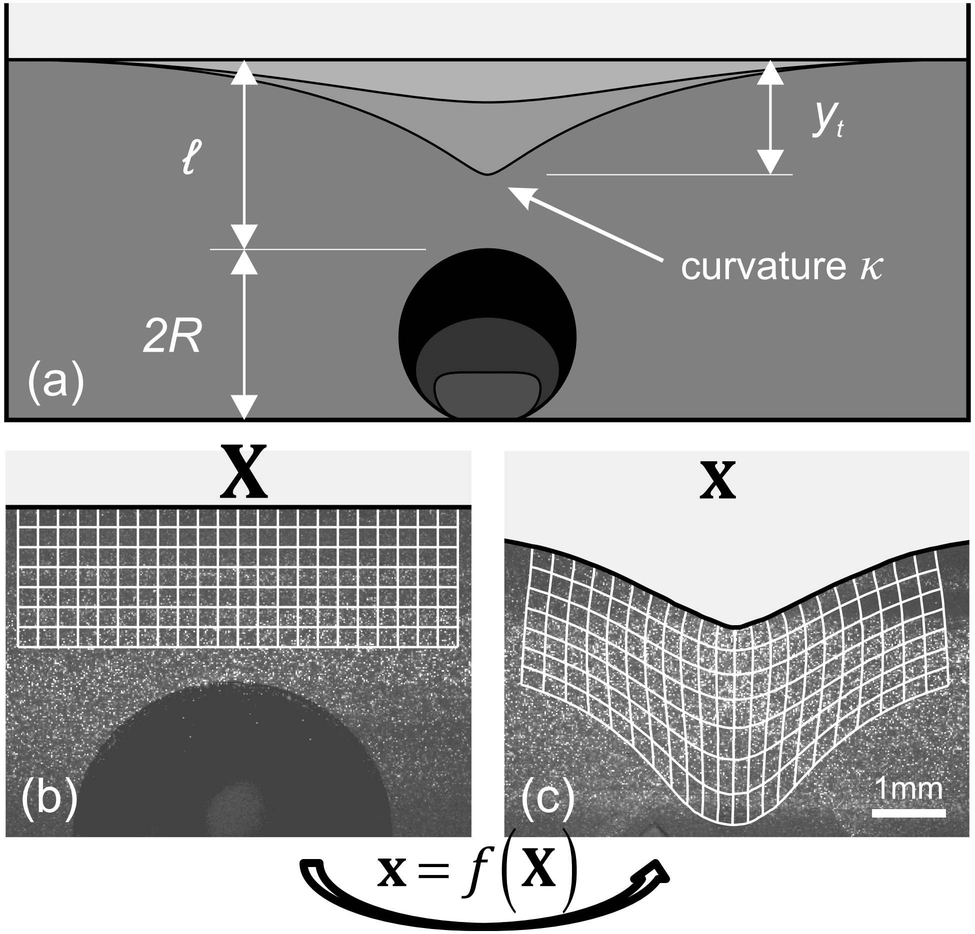
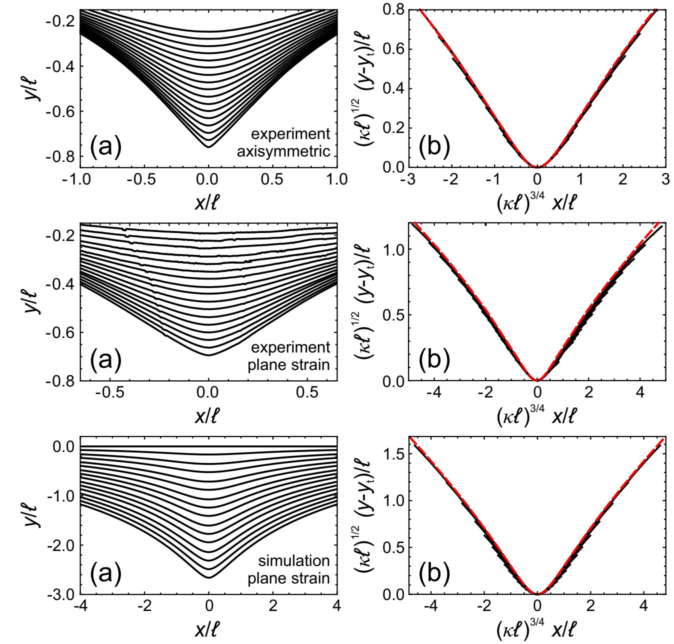
Elastic cusp

with S. Karpitschka, J. Snoeijer





$$x = \kappa^3\left(s + as^3\right) \qquad y = \kappa^2 s^2$$

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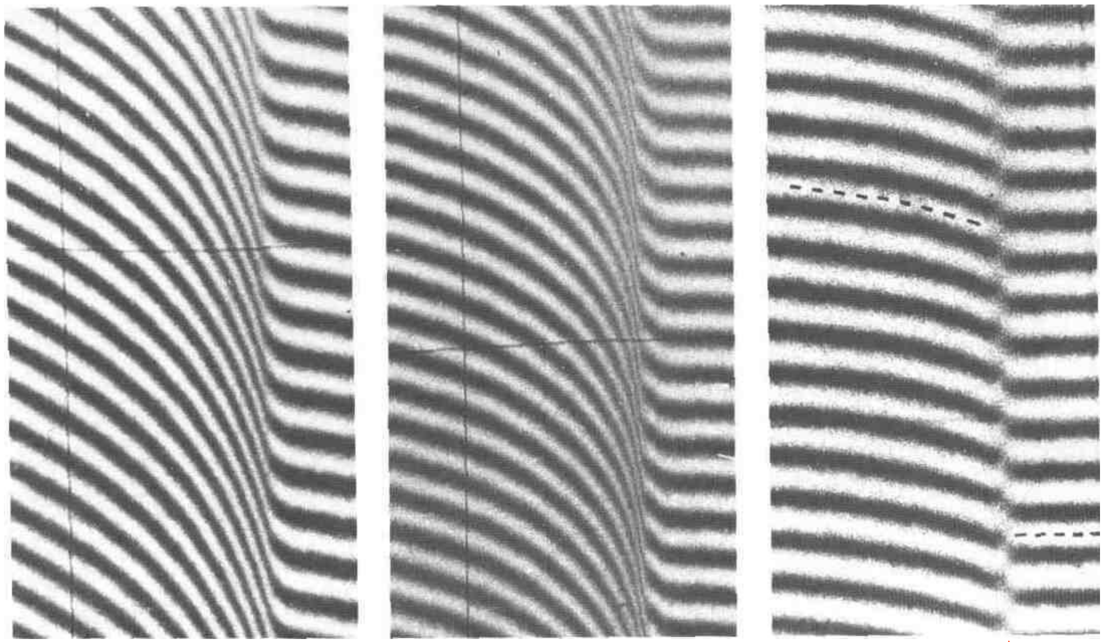
A shock wave



a jump in density occurs at some finite time t_0 !

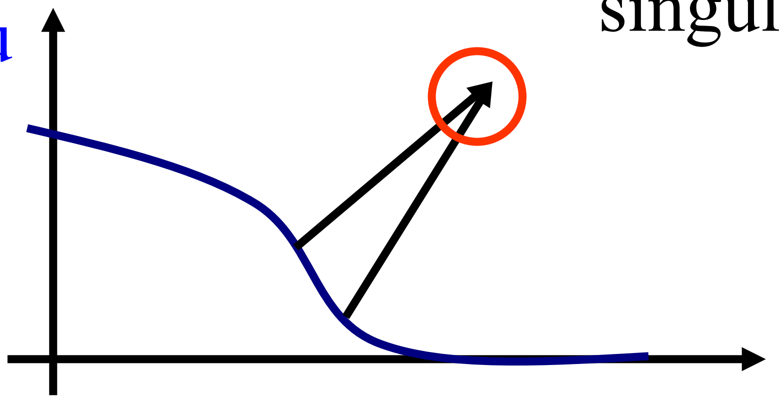
W.C. Griffith, W. Bleakney

Burgers' equation

characteristic curves:

$$x(\xi,t) = u_0(\xi)t + \xi$$

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0}$$



singularity in finite time

$$\boxed{u(x(\xi,t),t) = u_0(\xi)}$$

$$t_0 = Min_\xi \left\{ -1 \middle/ u_0'(\xi) \right\}$$

$$u_0(\xi) = -s' + as'^3 + \ldots$$

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Similarity solution

$$t' = t_0 - t$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

$$u = t'^{\alpha} U\left(\frac{x}{t'^{\alpha+1}}\right) \quad \xi = x / t'^{\alpha+1}$$

$$-\alpha U + (1+\alpha)\xi U' + UU' = 0$$

$$\xi = \begin{cases} -U - CU^{1+1/\alpha}, & \alpha_i = \dfrac{1}{2i+2}, i = 0,1,2,\ldots \\ -U, & \alpha = 0 \end{cases}$$

regular at $\xi = 0$

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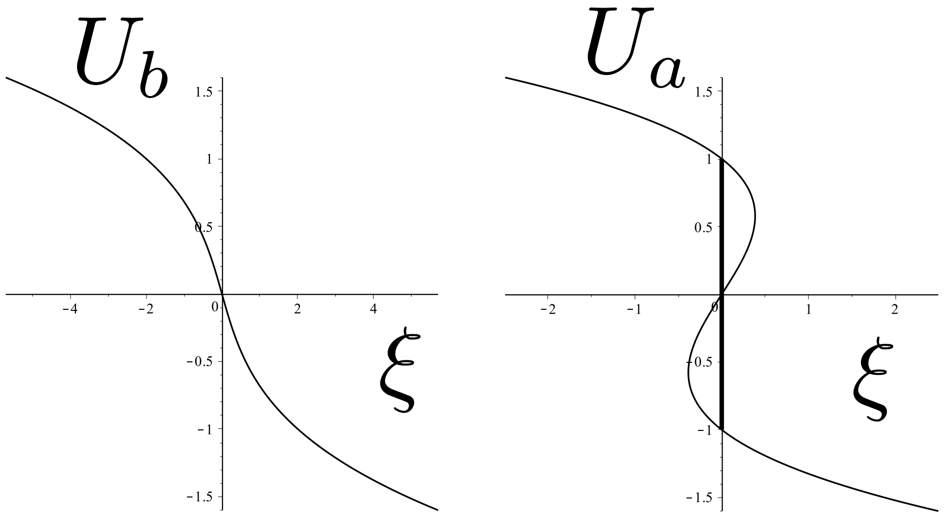
Similarity solution

$$t' = t_0 - t$$



$$\xi + U_b + C U_b^3 = 0$$

$$\xi - U_a + C U_a^3 = 0$$

$$\boxed{u(x,t) = t'^{1/2} U\left(x / t'^{3/2}\right)}$$

only stable solution!

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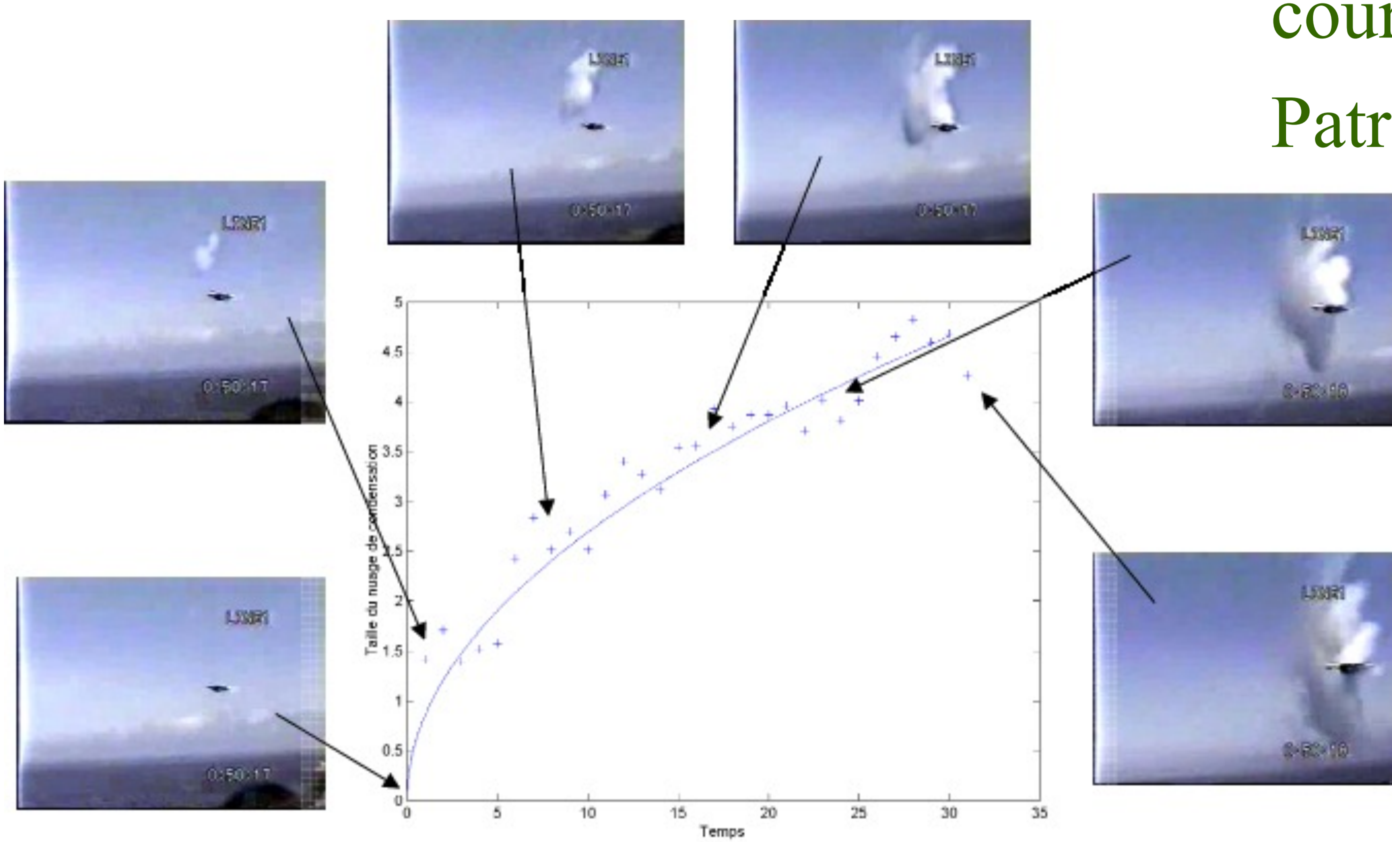
2D structure of shock waves

courtesy of
Patrice Legal



$$t_c(y) - t_0 = ay^2 + O(y^3),\ a > 0,\ y \sim t'^{1/2}$$

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Compressible Euler with T. Grava

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad p = \frac{A}{\gamma}\rho^{\gamma}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p \qquad \mathbf{v} = \nabla \phi$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 = -\frac{A}{\gamma - 1}\left(\rho^{\gamma-1} - \rho_0^{\gamma-1}\right)$$

$$|\nabla \rho|_{\rho=\rho_0} \to \infty$$

$$\partial \rho / \partial y = 0, \quad \partial x / \partial \rho = 0$$

$$\partial^2 x / \partial \rho^2 = 0, \quad \frac{\partial^3 x}{\partial \rho^3} = finite!$$

$$c_0^2 = \frac{\partial p}{\partial \rho} = A\rho_0^{\gamma-1}$$

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Similarity solution

$$\phi = |t'|^2\Phi(\xi,\eta), \quad \xi = \frac{x + c_0 t'}{|t'|^{3/2}}, \quad \eta = \frac{y}{|t'|^{1/2}}$$

$$\rho = \rho_0[1 + |t'|^{1/2}R(\xi,\eta) + |t'|Q(\xi,\eta)]$$

In(B):

$$-c_0|t'|^{1/2}\Phi_\xi + |t'|[\mp 2\Phi \pm \frac{3\xi}{2}\Phi_\xi \pm \frac{\eta}{2}\Phi_\eta] + \frac{|t'|}{2}\Phi_\xi^2 =$$

$$-\frac{c_0^2}{\gamma-1}\left\{(\gamma-1)|t'|^{1/2}R + |t'|[(\gamma-1)Q + \frac{1}{2}\left(\gamma^2 - 3\gamma + 2\right)R^2]\right\} + O(t'^{3/2})$$

In(C):

$$-c_0 R_\xi|t'|^{-1} + |t'|^{-1/2}[\mp\frac{R}{2} \pm \frac{3\xi}{2}R_\xi \pm \frac{\eta}{2}R_\eta - c_0 Q_\xi] +$$

$$\rho_0|t'|^{-1}\Phi_{\xi\xi} + |t'|^{-1/2}[\Phi_\xi R_\xi + \Phi_{\xi\xi}R] = O(t'^0)$$

$$U = \Phi_\xi$$

$$U - 3\xi U_\xi - \eta U_\eta = \pm(\gamma+1)UU_\xi$$

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Similarity solution

$$\xi(U,\eta): \boxed{\xi_U U - 3\xi + \eta\xi_\eta = \pm(\gamma+1)U}$$

$$\xi = \mp\frac{\gamma+1}{2}U - U^3 F\left(\frac{\eta}{U}\right)$$

regularity condition: $\left.\frac{\partial^3\xi}{\partial\eta^3}\right|_0 = -F'''(a) \stackrel{!}{=} const$

$$\xi = \frac{\gamma+1}{2}U - A_0U^3 - A_1U^2\eta - A_2U\eta^2 - A_3\eta^3$$

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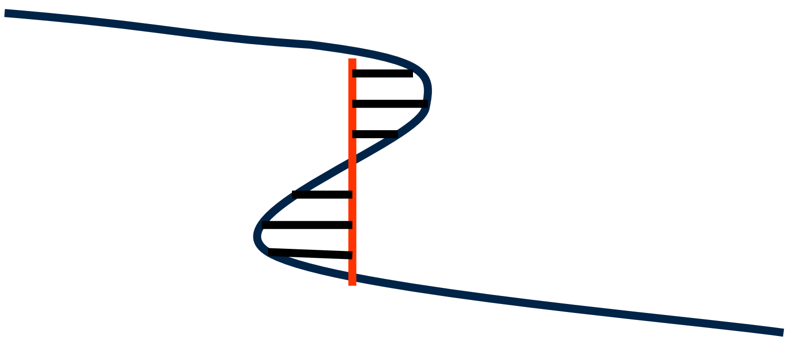
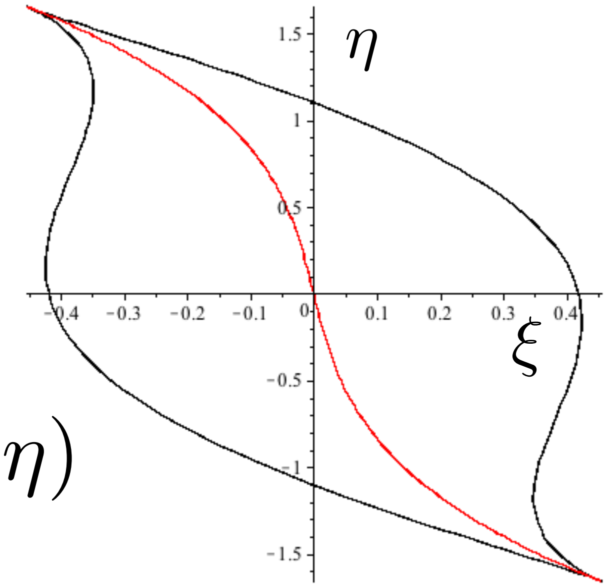
Shock position

$$\xi = \frac{\gamma+1}{2}U - A_0U^3 - A_1U^2\eta - A_2U\eta^2 - A_3\eta^3$$

$$U(\xi,\eta) \ \text{ vertical: } \frac{\gamma+1}{2} = 3A_0U^2 + 2A_1U\eta + A_2\eta^2$$





$$\bar{\xi} = \xi - \xi_s(\eta), \quad \bar{U} = U - U_s(\eta)$$

$$\bar{\xi} = -A_0\bar{U}\left(\bar{U}^2 - \Delta^2(\eta)\right)$$

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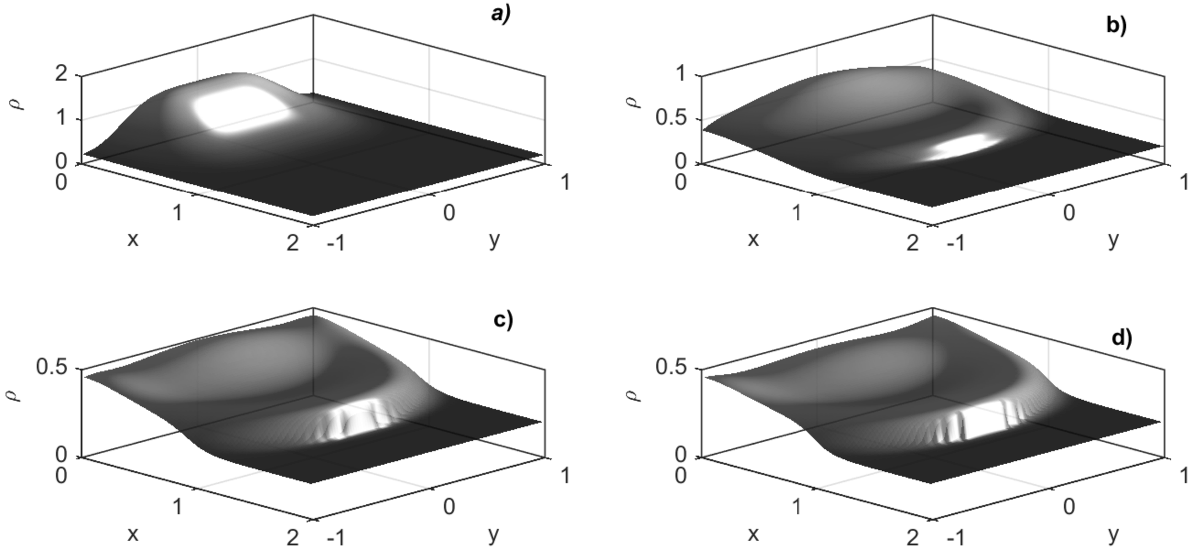
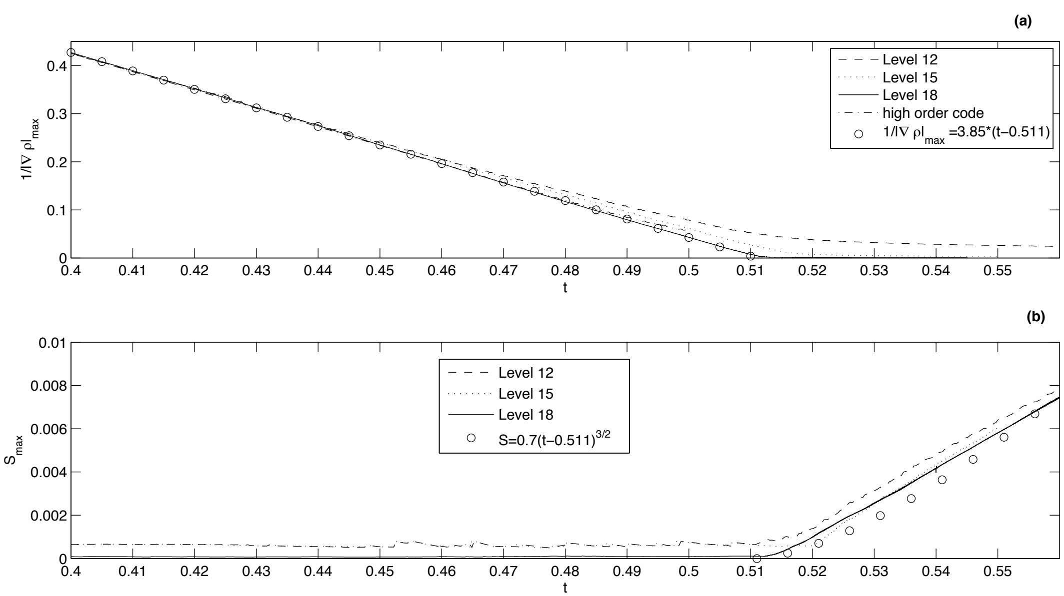
Numerical simulation





M.A. Herrada,
G. Pitton

with Basilisk

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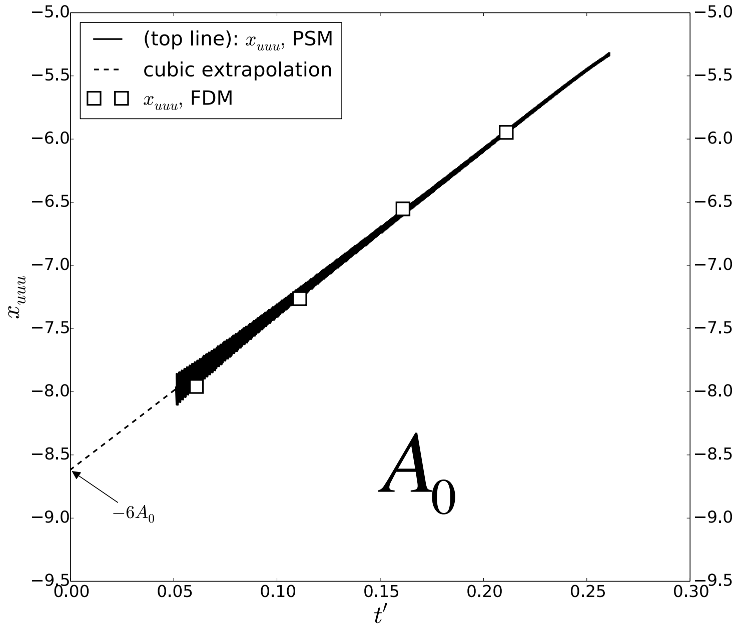
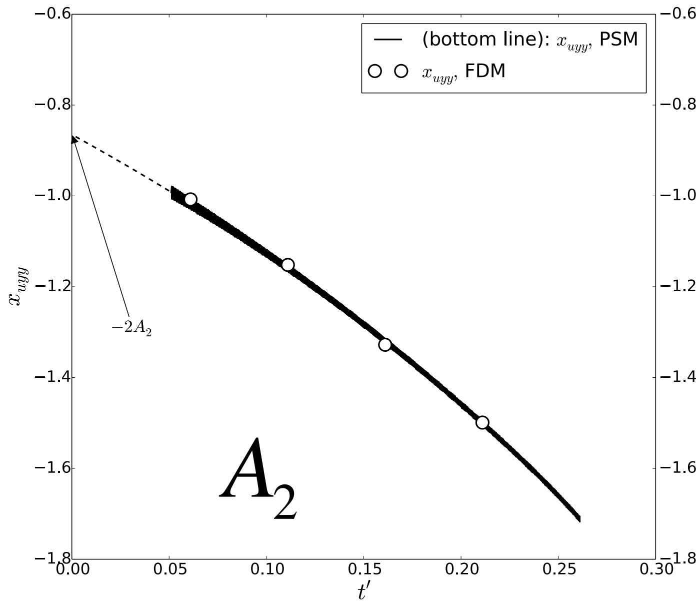
Parameters

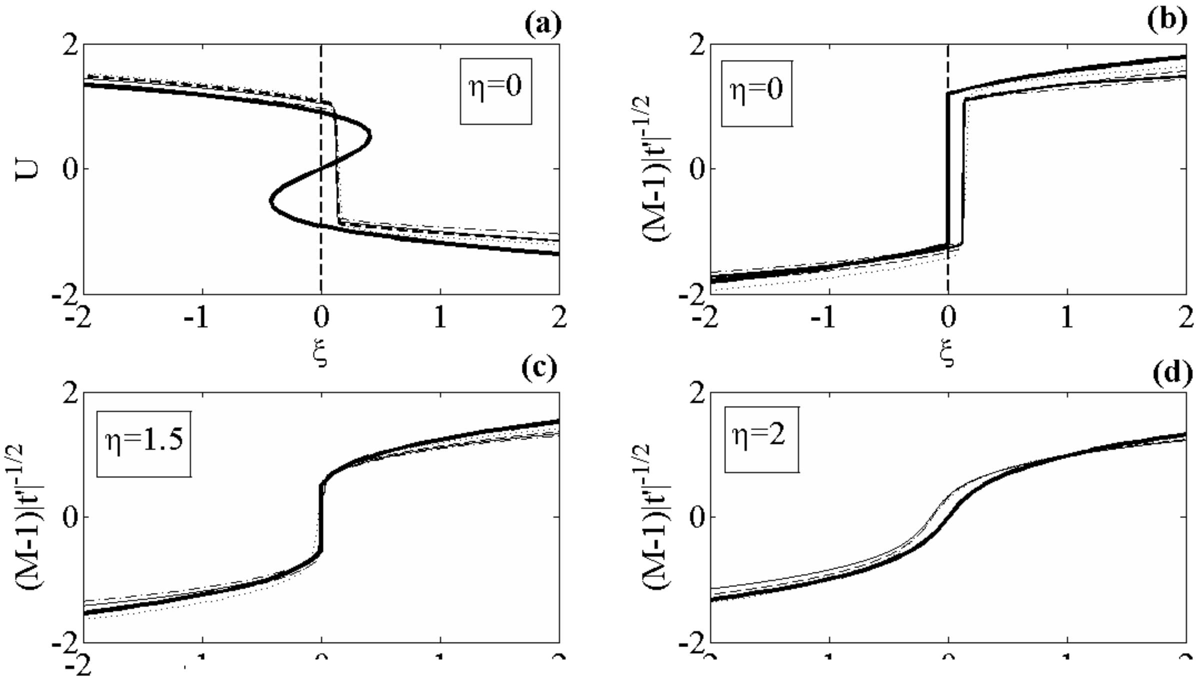




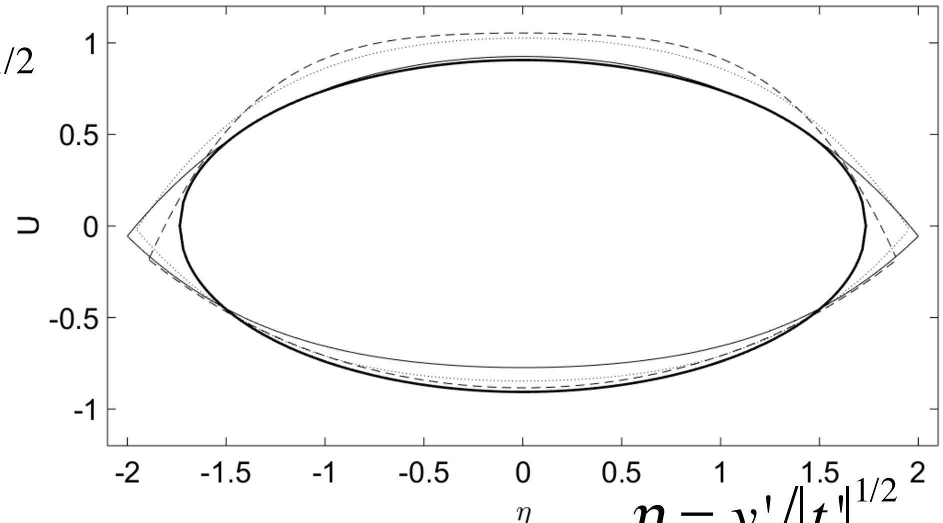
before singularity

$$\xi = \frac{\gamma + 1}{2} U - A_0 U^3 - A_1 U^2 \eta - A_2 U \eta^2 - A_3 \eta^3$$

Predictions after singularity



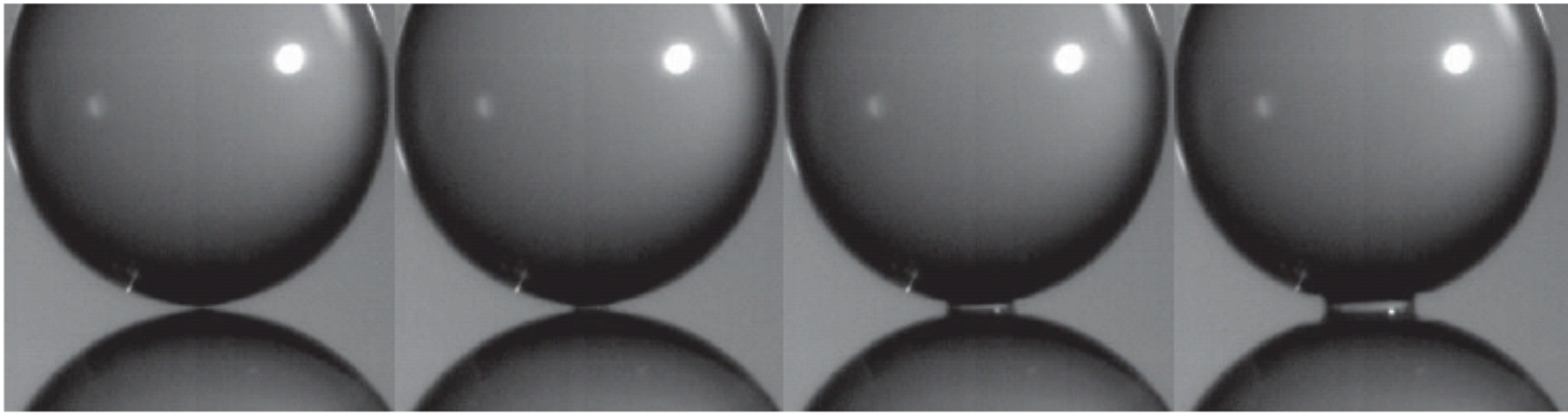
$$\xi = (x' - c|t'|)/|t'|^{3/2}$$



$$U = u/|t'|^{1/2}$$

$$\eta = y'/|t'|^{1/2}$$

Counterexample: drop coalescence



Aarts et al., PRL `05

width : r^2